Linear Approximation to Policy Function in IRIS Toolbox

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1 Introduction

This short note documents the solution of linear approximation to policy function of nonlinear dynamic stochastic general equilibrium model in the IRIS Toolobx 2006.12.02 by Jaromir Benes.

There are currently two options – one with unanticipated stochastic shocks with zero mean expectations and perfect foresight policy function.

2 Problem Formulation

A linear approximation to nonlinear model can be written as

$$F_1 \mathbb{E}_t x_{t+1} + F_2 x_t + F_3 \varepsilon_t + c = 0, \tag{1}$$

where the vector x stores transition variables of the model, ε denotes vector of stochastic shocks with $\mathbb{E}_t \varepsilon_{t+k} = 0$ for $k > 0$ unless perfect foresight solution is assumed.

In this note we abstract from formulation of the problem that allows for solution of non-stationary model and the determination of the constant or, in the sequel, of trends in series. Thus we impose for convenience $c = 0$ and vector x can be interpreted as deviation from a balanced growth path (BGP).

The vector x may be partitioned in the following form

$$x_{t+1} \equiv \begin{bmatrix} x_t^P \\ x_{t+1}^N \end{bmatrix}, \tag{2}$$

where x_t^P denotes predetermined and x_{t+1}^N non-predetermined transition variables.

Following Klein(2000), the policy function is solved using generalized Schur decomposition. However, definition of auxilliary processes is somewhat different than usual.

Let us define matrices Q, Z, A, B such that

$$QF_1Z = A \qquad QF_2Z = B, \tag{3}$$

where A, B are quasi-trianglular.

We define new auxilliary variables as follows

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} S_{t+1} \\ U_{t+1} \end{bmatrix} \equiv \begin{bmatrix} x_t^P \\ x_{t+1}^N \end{bmatrix}. \tag{4}$$

Using (4), we can rewrite (1) as follows

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} S_{t+1} \\ U_{t+1} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} S_t \\ U_t \end{bmatrix} + D\varepsilon_t = 0. \tag{5}$$

Due to properly reordered variables and Schur decomposition the lower part of (5) implies unstable dynamics.

2.1 Policy Function in Case of Unanticipated Shocks

2.1.1 Forward Solution of Unstable Part

Fist, we solve the unstable system forward and impose transversality condition. Thus we have

$$A_{22}\mathbb{E}_t U_{t+1} + B_{22}U_t + D_3\varepsilon_t = 0, \tag{6}$$

implying that

$$U_t = \lim_{k\to\infty} (-B_{22}^{-1}A_{22})^k \mathbb{E}_t U_{t+k} - \sum_{k=0}^{\infty} (-B_{22}^{-1}A_{22})^k B_{22}^{-1} D_2 \mathbb{E}_t \varepsilon_{t+k}. \tag{7}$$

Either directly from transversality conditiosn derived from optimization principles of the model or imposed directly – no-bubble solution is required, hence we postulate

$$\lim_{k\to\infty} (-B_{22}^{-1}A_{22})^k \mathbb{E}_t U_{t+k} = 0, \tag{8}$$

implying that

$$U_t = -\sum_{k=0}^{\infty} (-B_{22}^{-1}A_{22})^k B_{22}^{-1} D_2 \mathbb{E}_t \varepsilon_{t+k}. \tag{9}$$

Under our original assumption that $\mathbb{E}_t\varepsilon_{t+k} = 0$ for $k > 0$, the solution specializes to

$$U_t = -B_{22}^{-1} D_2 \varepsilon_t = R^U \varepsilon_t. \tag{10}$$

2.1.2 Solution to Stable Path

Having solution to U_t we can solve upper part of (5), hence we solve

$$A_{11}\mathbb{E}_t S_{t+1} + A_{12}U_{t+1} + B_{11}S_t + B_{12}U_t + D_1\varepsilon_t = 0. \tag{11}$$

Since $U_t = R^U\varepsilon_t$, then $U_{t+1} = R^U\varepsilon_{t+1}$ and $\mathbb{E}_t U_{t+1} = 0$.

After plugging these realations into (11), we obtain

$$A_{11}\mathbb{E}_t S_{t+1} = -B_{11}S_t - B_{12}R^U\varepsilon_t + D_1\varepsilon_t. \tag{12}$$

One has to realize that in general due to stochastic nature of the problem $\mathbb{E}_t S_{t+1} \neq S_{t+1}$, but we require solution in terms of S_{t+1} and S_t.

Using the fact that for predetermined variables we have $\mathbb{E}_t x_t^P - x_t^P = 0$, we can use (4) and write

$$\mathbb{E}_t \left(Z_{11} S_{t+1} + Z_{12} U_{t+1} \right) - \left(Z_{11} S_{t+1} + Z_{12} U_{t+1} \right) = 0, \tag{13}$$

implying that

$$\mathbb{E}_t S_{t+1} - S_{t+1} = Z_{11}^{-1} Z_{12} R^U \varepsilon_{t+1}. \tag{14}$$

Plugging (14) into (12) we obtain the solution

$$S_{t+1} = -A_{11}^{-1} B_{11} S_t - A_{11}^{-1} B_{12} R^U \varepsilon_t - A_{11}^{-1} D_1 \varepsilon_t - Z_{11}^{-1} Z_{12} R^U \varepsilon_{t+1}. \tag{15}$$

Using (4) we know that

$$S_{t+1} = Z_{11}^{-1} x_t^P - Z_{11}^{-1} Z_{12} U_{t+1} = Z_{11}^{-1} x_t^P - Z_{11}^{-1} Z_{12} R^U \varepsilon_{t+1} \tag{16}$$

Plugging (16) into solution (15) we obtain final stable solution

$$Z_{11}^{-1} x_t^P = -A_{11}^{-1} B_{11} Z_{11}^{-1} x_{t-1}^P - A_{11}^{-1} D_1 \varepsilon_t + \left(A_{11}^{-1} B_{11} Z_{11}^{-1} Z_{12} - A_{11}^{-1} B_{12} \right) R^U \varepsilon_t. \tag{17}$$

We define a new auxilliary variable α as

$$\alpha_t \equiv S_t + Z_{11}^{-1} Z_{12} U_t \equiv Z_{11}^{-1} x_t^P \tag{18}$$

and rewrite the stable transition equation (17) in terms of new variable α, i.e.

$$\alpha_t = -A_{11}^{-1} B_{11} \alpha_{t-1} - A_{11}^{-1} D_1 \varepsilon_t - \left(A_{11}^{-1} B_{11} G + A_{11}^{-1} B_{12} \right) R^U \varepsilon_t, \tag{19}$$

where $G = -Z_{11}^{-1} Z_{12}$.

To solve for x_t^N in terms of α, one must realize that $x_t^N = Z_{21} S_t + Z_{22} U_t$. We can thus rewrite

$$x_t^N = Z_{21} \left[Z_{11}^{-1} x_t^P - Z_{11}^{-1} Z_{12} U_t \right] + Z_{22} R^U \varepsilon_t. \tag{20}$$

hence

$$x_t^N = Z_{21} \alpha_{t-1} + \left[-Z_{21} Z_{11}^{-1} Z_{12} + Z_{22} \right] U_t \tag{21}$$

or

$$x_t^N = Z_{21} \alpha_{t-1} + \left[Z_{21} G + Z_{22} \right] U_t \tag{22}$$

2.1.3 State Space System with No Anticipations

The state-space model for *transition equations* thus may be written as follows

$$\begin{bmatrix} x_t^N \\ \alpha_t \end{bmatrix} = \begin{bmatrix} T^F \\ T^A \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} R^F \\ R^A \end{bmatrix} \varepsilon_t \tag{23}$$

$$x_t^P = U\alpha_t. \tag{24}$$

We define

$$T^F \equiv Z_{21} \tag{25}$$
$$T^A \equiv -A_{11}^{-1} B_{11} \tag{26}$$
$$R^F \equiv (Z_{21}G + Z_{22})\, R^U \tag{27}$$
$$R^A \equiv \left[-A_{11}^{-1} D_1 - \left(A_{11}^{-1} B_{11} G + A_{11}^{-1} B_{12}\right) R^U\right] \tag{28}$$
$$G \equiv -Z_{11}^{-1} Z_{12} \tag{29}$$
$$R^U \equiv -B_{22}^{-1} D_2 \tag{30}$$
$$U \equiv Z_{11} \tag{31}$$

2.2 Policy Function with Perfect Foresight

In case of *full perfect foresight* we solve the model (1) using assumption that

$$\mathbb{E}_t \varepsilon_{t+k} = \varepsilon_{t+k}, \qquad k > 0. \tag{32}$$

There are multiple ways how to solve for this case. Even an algorithm calculated under assumption of no foresight may be accomodated for inclusion of perfect foresight by appropriate use of auxilliary variables. One can then combine foresight with surprises.

For convenience we focus initially on pure foresight case, where there are no surprises and all shocks are perfectly anticipated. Furthermore, it is assummed that a steady-state of ε_{t+k} is zero. The solution used bellow cannot form an infinite sum of geometric series, i.e. 'proper' permament shock.

2.2.1 Forward Solution of Unstable Part

Starting from identical quasi-triangular decoupled system (5) we solve for the lower unstable part.

Imposing no-bubble equlibrium the solution collapses into

$$U_t = -\sum_{k=0}^{\infty} (-B_{22}^{-1} A_{22})^k B_{22}^{-1} D_2 \mathbb{E}_t \varepsilon_{t+k}. \tag{33}$$

For convenience let us define

$$J \equiv \left(-B_{22}^{-1} A_{22}\right) \qquad R^U \equiv (-B_{22} D_2), \tag{34}$$

hence simplifying the notation using

$$U_t = \sum_{k=0}^{\infty} J^k R^U \mathbb{E}_t \varepsilon_{t+k}. \tag{35}$$

It is clear that agents discount future shocks at the rate of J, which is formed using matrices of unstable dynamics. Intuitively, the model operates on unstable trajectory until the occurence of the shock anticipated. There are thus two effects of anticipated shocks (i) *announcement effect* and (ii) *implementation effect.*

For better intuition we shall use illustrative example, when agents anticipate only two periods ahead shocks and then expect zero realisations of shocks, i.e.

$$U_t = R^U \varepsilon_t + J R^U \varepsilon_{t+1} + J^2 R^U \varepsilon_{t+2}. \tag{36}$$

2.2.2 Solving Stable Part

Since we assume pure perfect foresight, it is clear that $\mathbb{E}_t S_{t+1} = S_{t+1}$ and we can thus write the stable part of the system as

$$A_{11} S_{t+1} + A_{12} U_{t+1} + B_{11} S_t + B_{12} U_t + D_1 \varepsilon_t = 0. \tag{37}$$

Using the definition of $\alpha \equiv S_t + Z_{11}^{-1} Z_{12} U_t$ we can rewrite the solution to the stable part of system as

$$\begin{aligned} \alpha_t = & - A_{11}^{-1} B_{11} \alpha_{t-1} \\ & + \left(Z_{11}^{-1} Z_{12} + A_{11}^{-1} A_{12}\right) U_{t+1} \\ & + \left(A_{11}^{-1} B_{11} Z_{11}^{-1} Z_{12} - A_{11}^{-1} B_{12}\right) U_t \\ & - A_{11}^{-1} D_1 \varepsilon_t. \end{aligned} \tag{38}$$

It should be clear that we can write a recursion for determination of U_t as

$$U_t = J U_{t+1} + R^U \varepsilon_t, \tag{39}$$

and rewrite (38) as

$$\begin{aligned} \alpha_t = & - A_{11}^{-1} B_{11} \alpha_{t-1} \\ & + \left(Z_{11}^{-1} Z_{12} + A_{11}^{-1} A_{12}\right) U_{t+1} \\ & + \left(A_{11}^{-1} B_{11} Z_{11}^{-1} Z_{12} - A_{11}^{-1} B_{12}\right) \left(J U_{t+1} + R^U \varepsilon_t\right) \\ & - A_{11}^{-1} D_1 \varepsilon_t, \end{aligned} \tag{40}$$

which is the final solution of the problem.

As in previous case we need to solve for x_t^N in terms of α. The procedure is identical, giving a solution

$$x_t^N = Z_{21} \alpha_{t-1} + \left[-Z_{21} Z_{11}^{-1} Z_{12} + Z_{22}\right] U_t. \tag{41}$$

2.2.3 Practical Implementation of the Solution

As in the previous case the solution for transition variables will take a state-space form.

The solution can be written down in a form

$$\begin{bmatrix} x_t^N \\ \alpha_t \end{bmatrix} = T \alpha_{t-1} + \mathbb{R} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t+1} \\ \vdots \\ \varepsilon_{t+N} \end{bmatrix}, \tag{42}$$

where $\mathbb{R}$ is *expanded* matrix of period t impact of current and anticipated shocks. The first left-block of $\mathbb{R}$ consists of stacked matrices $[R^F; R^A]$ capturing the effect of current period shock.

To write down the matrix $\mathbb{R}$ it is convenient to define auxilliary matrices. Thus, let

$$X^{A0} = \left(A_{11}^{-1} B_{11} Z_{11}^{-1} Z_{12} - A_{11}^{-1} B_{12}\right) \tag{43}$$

$$X^{A1} = \left(Z_{11}^{-1} Z_{12} + A_{11}^{-1} A_{12}\right) \tag{44}$$

$$X^{A} = X^{A1} + X^{A0} J \tag{45}$$

for the use with lower-part of the state-space and

$$X^F = (Z_{21}G + Z_{22}) \tag{46}$$

for the upper part.

The expanded matrix $\mathbb{R}$ can then be written as

$$\mathbb{R} = \begin{bmatrix} R^F & X^F R^U & X^F J R^U & X^F J^2 R^U & \dots & X^F J^{N-1} R^U \\ R^A & X^A R^U & X^A J R^U & X^A J^2 R^U & \dots & X^A J^{N-1} R^U \end{bmatrix} \tag{47}$$