Simulating Linear State-Space Models *

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Introduction

This simple note attempts to provide a brief explanation how one can simulate linear state-space models (DSGEs, VARs, ...) with exogenized/endogenized variables both with anticipations and without anticipations. These methods are used for producing forecasts using the 'G3' model of the Czech National Bank by Andrle, Kamenik, Vlcek and Hledik (2007).

Calculations are part of the IRIS Toolbox v. 2006-12-02 written by Jaromír Beneš for the Czech National Bank. Since there exists no documentation of the codes this short note is intended to facilitate communication of CNB's staff. Focus is on the *intuition*, please report errors.

State Space Form The state-space system used can be written as follows

$$\begin{bmatrix} x_t^N \\ \alpha_t \end{bmatrix} = \begin{bmatrix} T^F \\ T^A \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} R^F \\ R^A \end{bmatrix} \varepsilon_t \tag{1}$$

$$x_t^P = U\alpha_t \tag{2}$$

$$Y_t = Z\alpha_t + H\varepsilon_t, \tag{3}$$

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where x_t^N denotes $(nf \times 1)$ vector of non-predetermined transition variables. The $(nb \times 1)$ vector of *transformed* predetermined transition variables is denoted by α. Untransformed transition variables are put into vector x_t^P. Measurement equation (3) contains $(ny \times 1)$ vector of measurement variables, linked to transformed transition predetermined variables α_t. For convenience, let $T = [T^{F'}\ T^{A'}]'$.

The discussion above naturally holds for other linear state-space forms, different our particular one.

1 Definition of the Problem

Simulating the state-space model given the initial conditions for α_0 and trajectory $\{\varepsilon\}_1^T$ is straight-forward.

Bellow we demonstrate how it is possible to simulate the linear state-space model conditioned on particular trajectory of selected variables while assuming either unanticipated or fully anticipated shocks ε_t. Conditioning means that we specify values of selected variables at selected periods and choose some economic shocks at (possibly different) time periods to be calculated in a way to deliver required trajectory of selected variables.

Effectively, there are certain *exogenized* variables and corresponding *endogenized shocks (residuals)*. The only requirement for unique solution is that number of exogenized-variable-periods must be equal to endogenized-shock-periods. Thus, it is not necessary that a particular value of exogenized variable at time t must be delivered by endogenized shock at the same time!

Note, however, that the simulated path of all endogenous variables is dependent on the structure of exogenization, namely on choice of endogenized shocks and periods of endogenization.

If we setup a problem where one variable-period is for one shock-period, the solution is exact. Note that we have another option – to specify exogenous targets and calculate *the evolution of several structural shocks* that will deliver the solution. Since if the problem is predetermined there are multiple solution, following Sims and (?) we can choose the *most likely set of shocks.*

Following the terminology of Waggoner and Zha (1999) and Leeper and Zha (2003) we may distinguish *hard constraints/conditions/tunes* and *soft constraints/conditions/tunes.* This note is related to calculating hard tunes (fixes) since we specify the solution exactly, not as a possible range. To save on space we give explanation for calculating the basic hard fix with and without anticipated shoks.

1.1 Simulating with Anticipations – Reminder

In case of *perfect anticipation* of economic shocks we have shown elsewhere[1] how to expand the state-space system in order to allow for perfect-foresight simulation. For example, assume that last period of foreseen residual (i.e. different from zero in our case) is at $t+N$. Then expanded state-space of the form

$$\begin{bmatrix} x_t^N \\ \alpha_t \end{bmatrix} = \begin{bmatrix} T^F \\ T^A \end{bmatrix} \alpha_{t-1} + \mathbb{R} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t+1} \\ \dots \\ \varepsilon_{t+N} \end{bmatrix}, \tag{4}$$

where $\mathbb{R}$ is defined as

$$\mathbb{R} = \begin{bmatrix} R^F & X^F R^U & X^F J R^U & \dots & X^F J^{N-1} R^U \\ R^A & X^A R^U & X^A J R^U & \dots & X^A J^{N-1} R^U \end{bmatrix}, \tag{5}$$

where X^F, X^A and J, R^U are defined in Andrle(2007). It is convenient to define

$$\mathbb{R}_N \equiv \begin{bmatrix} \mathbb{R}^F \\ \mathbb{R}^A \end{bmatrix} \equiv \begin{bmatrix} \mathbb{R}_1^F & \mathbb{R}_2^F & \dots & \mathbb{R}_N^F \\ \mathbb{R}_1^A & \mathbb{R}_2^A & \dots & \mathbb{R}_N^A \end{bmatrix}, \tag{6}$$

as partitioned matrix corresponding both to predetermined and nonpredetermined transition variables and to particular time periods of forward anticipation. We add subindex N to $\mathbb{R}$ to denote the number of column-partitions of the expanded matrix.

[1] Andrle, M.: Linear Approximation to Policy Function in Iris Toolbox, July 2007

2 Simulating with Exogenized and Endogenized Variables

Assume we have chosen values of particular variables at particular time periods. These can be achieved by selecting particular shocks in particular periods appropriately.

The problem is then to calculate values for residuals (shocks) that satisfy the constraint. The principle is intuitive. By simulating forward the model (or expanded model) we can exactly trace the *impact* of particular shock $\varepsilon_{i,t}$ on all variables. Thus, we will calculate so called *multiplier matrices* or impact matrices. These are in principle impulse-response matrices.

Knowing the multiplier matrices, we can proceed as follows. Given non-endogenized shocks, we simulate the *free solution* of the model, implied by initial conditions and exogenous shocks. Unless by chance, the solution will not be equal to our desired path for exogenized variables. We need to find such values of endogenized shocks that would eliminate the discrepancy. Due to linearity of the model, this is just a linear system of equations problem.

A simple example may ehnance the intuition perhaps.

EXAMPLE: Let us have a simple model with perfectly anticipated shocks. Let us assume that last nonzero shock is for $t = 3$, hence it is sufficient to expand the state-space for three periods only. We can stack the solution of the problem and calculate impact matrices.

$$x_1 = Tx_0 + [R_1 \ R_2 \ R_3] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{7}$$

$$x_2 = Tx_1 + [R_1 \ R_2] \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \tag{8}$$

$$x_3 = Tx_2 + R_1\varepsilon_3. \tag{9}$$

For all simulated periods $t = 1, 2, 3$ we can stack the solution and find out the multipliers of particular shocks.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 & R_3 \\ TR_1 & (TR_2 + R_1) & (TR_3 + R_2) \\ T^2R_1 & T(TR_2 + R_1) & T(TR_3 + R_2) + R_1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} + \begin{bmatrix} T \\ T^2 \\ T^3 \end{bmatrix} x_0 \quad (10)$$

where the left-hand side matrix is $(3nx \times 1)$, the vector of errors is $(3ne \times 1)$ and the overall impact matrix is $(3nx \times 3ne)$ in our simple example.

For simplicity of notation, let us assume that $nx = ne$. Assume that we require vector x_3 (i.e. all variables in the third period) to be of some specific value, say $x_3 = x_3^{fix}$. Given values for all shocks in period 2 and 3 and initial conditions expressed by x_0, what is the value of ε_1 for the model to meet the restriction?

First, simulate the model with given $\varepsilon_2, \varepsilon_3$ and $\varepsilon_1 = 0$. Denote the result of free simulation as x^{free}. Define $\xi = x_3^{fix} - x_3^{free}$ as the discrepancy of the required solution from the free solution. We need to find a value to be added to our ε_3 that would deliver required solution, i.e.

$$(x_3^{free} - x_3^{fix}) = T^2R_1(\varepsilon_1^{old} - \varepsilon_1^{new}) \quad (11)$$

which is just a simple linear problem with a solution

$$\left(\varepsilon_1^{old} - \varepsilon_1^{new}\right) = \left(T^2R_1\right)^{-1}(x_3^{free} - x_3^{fix}) \quad (12)$$

Using the new value for ε_3 to simulate the model changes the solution of variables that were not exogenized, but the condition for x_3 will be met.

2.1 No Anticipation, Balanced-time Fixing

In case of no anticipation of structural shocks and when we are fixing the same number of variables as we are endogenizing *at each particular time period*, the situation is much easier than in other cases.

There are two main reasons why the situation is simple. First, since shocks are unanticipated, they matter for the dynamics only in the period they come and in periods onward. It is thus very easy to trace the *impact* of the shock on all variables in linear case.

Second, since at each period in which there is some fixed variable, we also have some endogenized variable. We can thus proceed step-by-step and set the endogenous residual ε_t in such a way that it will deliver the required value of fixed variable – x_t^N, α_t or Y_t.

Let us define the *instant multiplier* or *instant impact* matrix for the vector $x_t = [x^{N'} \ \alpha']$ and for measurment variables Y_t as

$$MX = \begin{bmatrix} MX_1 \\ MX_2 \end{bmatrix} = \begin{bmatrix} R^F \\ R^A \end{bmatrix} \qquad MY = Z(MX_2) + H. \tag{13}$$

Then we can proceed as follows. Given α_0 (the initial condition for state variables) and trajectory of exogenous residuals $\{\varepsilon_t\}_0^T$. At each time, we calculate the *free solution* and contrast value of exogenized variables to desired values. Conditioning on instant impact matrices MX and MY we find out value of endogenized partition of ε_t such that constraints on exogenous variables are met.

New values of endogenized shocks will propagate into next periods by means of predetermined transition variable α. But since in the next period we have both exogenized and corresponding amount of endogenized variables (shocks), we repeat the problem.

Going back to the simple example above, we can see that in case of anticipated shocks and/or unbalanced-time fixing –which are both present in the example– we would not be able to use this simple step-by-step solution method. In case of anticipations – the value of shock at $t + N$ affects solution already at t, hence the whole problem must be stacked into one large problem.

2.2 Unbalanced-time Fixing and/or Anticipated Shocks

When anticipated shocks are assumed and/or mixed with unbalanced-time fixing of variables, we proceed basically as in the example above. We form large stacked linear problem where we map all shocks to all variables.

The basic form of the problem is as follows

$$\begin{bmatrix} x_t \\ x_{t+1} \\ \dots \\ x_{t+K} \end{bmatrix} = \begin{bmatrix} MX_{1,\varepsilon_1} & \dots & MX_{\varepsilon_{t+N}} \\ MX_{1,\varepsilon_1} & \dots & MX_{\varepsilon_{t+N}} \\ \dots & \dots & \dots \\ MX_{1,\varepsilon_1} & \dots & MX_{\varepsilon_{t+N}} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dots \\ \varepsilon_N \end{bmatrix}, \tag{14}$$

where the index K goes towards the last exogenous variable in the time span and N goes towards the last endogenized residual in the model.

The MX multiplier matrix is of the $(nxK \times neN)$ dimension. Note that $nx = nf + nb$, since $x_t = [x^{N'} \; \alpha']'$. Note that the first 'partition-row' of impact matrix of dimension $(nx \times neN)$ for the first period $t = 1$ is identical to expanded matrix $\mathbb{R}_N$ expanded N periods forward. When the expansion of the state-space for anticipated shocks is carried out, is equal to greater value from last observed residual (LR) and last endogenized variable N.

Then, we recursively update the 'partition rows' of the multiplier matrix MX. The first 'partition row' (with initial conditions) gives arise to x_1, which is then recursively propagated to x_2 by T as cen be seen for instance in the simple example above. The vector of shocks from $t = 1$ period is thus carried only through past effect through T. However shocks from $t = 2, \dots N$ are still in the instantanous impact matrix for time $t = 2$ which is $\mathbb{R}_{N-1}$. The subindex $N - 1$ means that the original matrix $\mathbb{R}_N$ which is $(nx \times neN)$ is now reduced to $(nx \times ne(N - 1))$ by cuting the leftmost part of the expanded matrix. In `Matlab` notation we can write that `RR2 = RR1(:,ne*(N-1))`. In this way the $\mathbb{R}$ matrix is reduced period-by-period. At the last period where no foreseeable shocks are byond the period the $\mathbb{R}_{N-N-1} = R$.

In principle each nest 'partition-row' is the previous row multiplied by matrix T (the transition/update move) and for 'partition columns' corresponding to current and onwards periods the instatnous matrix $\mathbb{R}_t$ is added. The pattern is best viewed in the simple example above where always only to columns on and above diagonal the current impact matrix is added.

In the `Matlab` code of the IRIS toolbox the simplified version may be run as follows: First, we initialize the impact matrix $\mathbb{R}$ for the case of anticipations or no anticipations. In case of anticipations the columns are expanded to treat all shocks forward up to last endogenous shock.

```
if anticipate == true
  RR = R(:, 1:ne*lastendog);
  MX(1:nx, :) = RR;
else
  RR = R(:, 1:ne);
  MX(1:nx, 1:ne) = RR;
end
```

Then we run the recusion and at each particular time we fill the matrix MX and the matrix MY by the updating scheme, i.e. multiply by T and if column above the 'partitioned diagonal', add the current impact matrix.

We also need to form multiplier matrix for transition variables in Y. Calculation of MY matrix is straight-forward, since we have that for each 'partition-row' (i.e. period)

$$MY_t = Z(MX_2)_t + H. \tag{15}$$

At the end, we get two impact-multiplier matrices MX and MY mapping shocks into transtion and measurement variables through a simple system of linear equations.