Introduction

This simple note attempts to provide a brief explanation how one can simulate linear state-space models (DSGEs, VARs, ...) with exogenized/endogenized variables both with anticipations and without anticipations. These methods are used for producing forecasts using the 'G3' model of the Czech National Bank by Andrle, Kamenik, Vlcek and Hledik (2007).

Calculations are part of the IRIS Toolbox v. 2006-12-02 written by Jaromír Beneš for the Czech National Bank. Since there exists no documentation of the codes this short note is intended to facilitate communication of CNB’s staff. Focus is on the intuition, please report errors.

State Space Form

The state-space system used can be written as follows

\[
\begin{align*}
\begin{bmatrix} x_t^N \\ \alpha_t \end{bmatrix} &= \begin{bmatrix} T^F \\ T^A \end{bmatrix} \alpha_{t-1} + \begin{bmatrix} R^F \\ R^A \end{bmatrix} \varepsilon_t \\
x_t^P &= U \alpha_t \\
Y_t &= Z \alpha_t + H \varepsilon_t,
\end{align*}
\]  

(1)

(2)

(3)
where \( x^N_t \) denotes \((nf \times 1)\) vector of non-predetermined transition variables. The \((nb \times 1)\) vector of transformed predetermined transition variables is denoted by \( \alpha \). Untransformed transition variables are put into vector \( x^P_t \). Measurement equation (3) contains \((ny \times 1)\) vector of measurement variables, linked to transformed transition predetermined variables \( \alpha_t \). For convenience, let 
\[
T = [T^F' \ T^A']'.
\]

The discussion above naturally holds for other linear state-space forms, different our particular one.

1 Definition of the Problem

Simulating the state-space model given the initial conditions for \( \alpha_0 \) and trajectory \( \{ \varepsilon \}^T_1 \) is straight-forward.

Below we demonstrate how it is possible to simulate the linear state-space model conditioned on particular trajectory of selected variables while assuming either unanticipated or fully anticipated shocks \( \varepsilon_t \). Conditioning means that we specify values of selected variables at selected periods and choose some economic shocks at (possibly different) time periods to be calculated in a way to deliver required trajectory of selected variables.

Effectively, there are certain exogenized variables and corresponding endogenized shocks (residuals). The only requirement for unique solution is that number of exogenized-variable-periods must be equal to endogenized-shock-periods. Thus, it is not necessary that a particular value of exogenized variable at time \( t \) must be delivered by endogenized shock at the same time!

Note, however, that the simulated path of all endogenous variables is dependent on the structure of exogenization, namely on choice of endogenized shocks and periods of endogenization.

If we setup a problem where one variable-period is for one shock-period, the solution is exact. Note that we have another option – to specify exogenous targets and calculate the evolution of several structural shocks that will deliver the solution. Since if the problem is predetermined there are multiple solution, following Sims and (?) we can choose the most likely set of shocks.
Following the terminology of Waggoner and Zha (1999) and Leeper and Zha (2003) we may distinguish hard constraints/conditions/tunes and soft constraints/conditions/tunes. This note is related to calculating hard tunes (fixes) since we specify the solution exactly, not as a possible range. To save on space we give explanation for calculating the basic hard fix with and without anticipated shocks.

1.1 Simulating with Anticipations – Reminder

In case of perfect anticipation of economic shocks we have shown elsewhere\(^1\) how to expand the state-space system in order to allow for perfect-foresight simulation. For example, assume that last period of foreseen residual (i.e. different from zero in our case) is at \(t + N\). Then expanded state-space of the form

\[
\begin{bmatrix}
    x_t^N \\
    \alpha_t
\end{bmatrix} = 
\begin{bmatrix}
    T^F \\
    T^A
\end{bmatrix} \alpha_{t-1} + \mathbb{R} 
\begin{bmatrix}
    \varepsilon_t \\
    \varepsilon_{t+1} \\
    \vdots \\
    \varepsilon_{t+N}
\end{bmatrix},
\]

where \(\mathbb{R}\) is defined as

\[
\mathbb{R} = 
\begin{bmatrix}
    R^F & X^F R^U & X^F J R^U & \cdots & X^F J^{N-1} R^U \\
    R^A & X^A R^U & X^A J R^U & \cdots & X^A J^{N-1} R^U
\end{bmatrix},
\]

where \(X^F, X^A\) and \(J, R^U\) are defined in Andrle(2007). It is convenient to define

\[
\mathbb{R}_N \equiv 
\begin{bmatrix}
    \mathbb{R}^F \\
    \mathbb{R}^A
\end{bmatrix} \equiv 
\begin{bmatrix}
    \mathbb{R}_1^F & \mathbb{R}_2^F & \cdots & \mathbb{R}_N^F \\
    \mathbb{R}_1^A & \mathbb{R}_2^A & \cdots & \mathbb{R}_N^A
\end{bmatrix},
\]

as partitioned matrix corresponding both to predetermined and nonpredetermined transition variables and to particular time periods of forward anticipation. We add subindex \(N\) to \(\mathbb{R}\) to denote the number of column-partitions of the expanded matrix.

\(^1\)Andrle, M.: Linear Approximation to Policy Function in Iris Toolbox, July 2007
Simulating Linear State-Space Models

2 Simulating with Exogenized and Endogenized Variables

Assume we have chosen values of particular variables at particular time periods. These can be achieved by selecting particular shocks in particular periods appropriately.

The problem is then to calculate values for residuals (shocks) that satisfy the constraint. The principle is intuitive. By simulating forward the model (or expanded model) we can exactly trace the impact of particular shock $\varepsilon_{i,t}$ on all variables. Thus, we will calculate so called multiplier matrices or impact matrices. These are in principle impulse-response matrices.

Knowing the multiplier matrices, we can proceed as follows. Given non-endogenized shocks, we simulate the free solution of the model, implied by initial conditions and exogenous shocks. Unless by chance, the solution will not be equal to our desired path for exogenized variables. We need to find such values of endogenized shocks that would eliminate the discrepancy. Due to linearity of the model, this is just a linear system of equations problem.

A simple example may enhance the intuition perhaps.

**EXAMPLE:** Let us have a simple model with perfectly anticipated shocks. Let us assume that last nonzero shock is for $t = 3$, hence it is sufficient to expand the state-space for three periods only. We can stack the solution of the problem and calculate impact matrices.

\[
\begin{align*}
x_1 &= Tx_0 + [R_1 \ R_2 \ R_3] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\
x_2 &= Tx_1 + [R_1 \ R_2] \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\
x_3 &= Tx_2 + R_1 \varepsilon_3.
\end{align*}
\]

For all simulated periods $t = 1, 2, 3$ we can stack the solution and find out the multipliers of particular shocks.
\[
\begin{pmatrix}
  x_1 \\ x_2 \\ x_3 
\end{pmatrix}
= \begin{pmatrix}
  R_1 & R_2 & R_3 \\ TR_1 & (TR_2 + R_1) & (TR_3 + R_2) \\ T^2R_1 & T(TR_2 + R_1) & T(TR_3 + R_2) + R_1 
\end{pmatrix}
\begin{pmatrix}
  \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 
\end{pmatrix}
+ \begin{pmatrix}
  T \\ T^2 \\ T^3 
\end{pmatrix} x_0
\] (10)

where the left-hand side matrix is \((3nx \times 1)\), the vector of errors is \((3ne \times 1)\) and the overall impact matrix is \((3nx \times 3ne)\) in our simple example.

For simplicity of notation, let us assume that \(nx = ne\). Assume that we require vector \(x_3\) (i.e. all variables in the third period) to be of some specific value, say \(x_3 = x_3^{fix}\). Given values for all shocks in period 2 and 3 and initial conditions expressed by \(x_0\), what is the value of \(\varepsilon_1\) for the model to meet the restriction?

First, simulate the model with given \(\varepsilon_2, \varepsilon_3\) and \(\varepsilon_1 = 0\). Denote the result of free simulation as \(x_3^{free}\). Define \(\xi = x_3^{fix} - x_3^{free}\) as the discrepancy of the required solution from the free solution. We need to find a value to be added to our \(\varepsilon_3\) that would deliver required solution, i.e.

\[
(x_3^{free} - x_3^{fix}) = T^2R_1(\varepsilon_1^{old} - \varepsilon_1^{new})
\] (11)

which is just a simple linear problem with a solution

\[
(\varepsilon_1^{old} - \varepsilon_1^{new}) = (T^2R_1)^{-1} (x_3^{free} - x_3^{fix})
\] (12)

Using the new value for \(\varepsilon_3\) to simulate the model changes the solution of variables that were not exogenized, but the condition for \(x_3\) will be met.

### 2.1 No Anticipation, Balanced-time Fixing

In case of no anticipation of structural shocks and when we are fixing the same number of variables as we are endogenizing at each particular time period, the situation is much easier than in other cases.

There are two main reasons why the situation is simple. First, since shocks are unanticipated, they matter for the dynamics only in the period they come and in periods onward. It is thus very easy to trace the impact of the shock on
all variables in linear case.

Second, since at each period in which there is some fixed variable, we also have some endogenized variable. We can thus proceed step-by-step and set the endogenous residual $\varepsilon_t$ in such a way that it will deliver the required value of fixed variable – $x_t^N, \alpha_t$ or $Y_t$.

Let us define the instant multiplier or instant impact matrix for the vector $x_t = [x^N, \alpha']$ and for measurement variables $Y_t$ as

$$MX = \begin{bmatrix} MX_1 \\ MX_2 \end{bmatrix} = \begin{bmatrix} R^T \\ R^A \end{bmatrix}$$

$$MY = Z(MX_2) + H. \tag{13}$$

Then we can proceed as follows. Given $\alpha_0$ (the initial condition for state variables) and trajectory of exogenous residuals $\{\varepsilon_t\}_0^T$. At each time, we calculate the free solution and contrast value of exogenized variables to desired values. Conditioning on instant impact matrices $MX$ and $MY$ we find out value of endogenized partition of $\varepsilon_t$ such that constraints on exogenous variables are met.

New values of endogenized shocks will propagate into next periods by means of predetermined transition variable $\alpha$. But since in the next period we have both exogenized and corresponding amount of endogenized variables (shocks), we repeat the problem.

Going back to the simple example above, we can see that in case of anticipated shocks and/or unbalanced-time fixing – which are both present in the example – we would not be able to use this simple step-by-step solution method. In case of anticipations – the value of shock at $t + N$ affects solution already at $t$, hence the whole problem must be stacked into one large problem.

### 2.2 Unbalanced-time Fixing and/or Anticipated Shocks

When anticipated shocks are assumed and/or mixed with unbalanced-time fixing of variables, we proceed basically as in the example above. We form large stacked linear problem where we map all shocks to all variables.
The basic form of the problem is as follows

$$
\begin{bmatrix}
  x_t \\
  x_{t+1} \\
  \vdots \\
  x_{t+K}
\end{bmatrix}
= \begin{bmatrix}
  MX_{1,\epsilon_1} & \ldots & MX_{\epsilon_{t+1},N} \\
  \ldots & \ldots & \ldots \\
  MX_{1,\epsilon_1} & \ldots & MX_{\epsilon_{t+1},N}
\end{bmatrix}
\begin{bmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \vdots \\
  \epsilon_N
\end{bmatrix},
$$

(14)

where the index $K$ goes towards the last exogenous variable in the time span and $N$ goes towards the last endogenized residual in the model.

The $MX$ multiplier matrix is of the $(nxK \times neN)$ dimension. Note that $nx = nf + nb$, since $x_t = [x^N \alpha']$. Note that the first 'partition-row' of impact matrix of dimension $(nx \times neN)$ for the first period $t = 1$ is identical to expanded matrix $R_N$ expanded $N$ periods forward. When the expansion of the state-space for anticipated shocks is carried out, is equal to greater value from last observed residual ($LR$) and last endogenized variable $N$.

Then, we recursively update the 'partition rows' of the multiplier matrix $MX$. The first 'partition row' (with initial conditions) gives arise to $x_1$, which is then recursively propagated to $x_2$ by $T$ as can be seen for instance in the simple example above. The vector of shocks from $t = 1$ period is thus carried only through past effect through $T$. However shocks from $t = 2, \ldots N$ are still in the instantaneous impact matrix for time $t = 2$ which is $R_{N-1}$. The subindex $N - 1$ means that the original matrix $R_N$ which is $(nx \times neN)$ is now reduced to $(nx \times ne(N - 1))$ by cutting the leftmost part of the expanded matrix. In Matlab notation we can write that $RR2 = RR1(:,ne*(N-1))$. In this way the $R$ matrix is reduced period-by-period. At the last period where no foreseeable shocks are beyond the period the $R_{N-N-1} = R$.

In principle each nest 'partition-row' is the previous row multiplied by matrix $T$ (the transition/update move) and for 'partition columns' corresponding to current and onwards periods the instantaneous matrix $R_t$ is added. The pattern is best viewed in the simple example above where always only to columns on and above diagonal the current impact matrix is added.

In the Matlab code of the IRIS toolbox the simplified version may be run as follows: First, we initialize the impact matrix $R$ for the case of anticipations or no anticipations. In case of anticipations the columns are expanded to treat all
shocks forward up to last endogenous shock.

```matlab
if anticipate == true
    RR = R(:, 1:ne*lastendog);
    MX(1:nx, :) = RR;
else
    RR = R(:, 1:ne);
    MX(1:nx, 1:ne) = RR;
end
```

Then we run the recursion and at each particular time we fill the matrix $MX$ and the matrix $MY$ by the updating scheme, i.e. multiply by $T$ and if column above the ‘partitioned diagonal’, add the current impact matrix.

We also need to form multiplier matrix for transition variables in $Y$. Calculation of $MY$ matrix is straight-forward, since we have that for each ‘partition-row’ (i.e. period)

$$MY_t = Z(MX_t) + H. \quad (15)$$

At the end, we get two impact-multiplier matrices $MX$ and $MY$ mapping shocks into transition and measurement variables through a simple system of linear equations.