Kalman Filtering and Smooting
in the IRIS Toolbox 2006.12.02

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Introduction

This simple note attempts to provide a brief documentation of the Kalman filter and smoother procedures in the IRIS Toolbox ver. 2006.12.02, written by Jaromir Benes.

In addition we provide a simple and useful extension to calculate weights for the Kalman filter and smoother.

IRIS State Space The state-space system used can be written as follows

$$\begin{aligned} Y_t &= Z\alpha_t + D + H\varepsilon_t && (1)\\ \alpha_t &= T\alpha_t + K + R\varepsilon_t && (2)\\ \mathbf{X_t} &= \mathbb{U}\alpha_t, && (3) \end{aligned}$$

where Y_t is a $(ny \times 1)$ vector of measurement variables, α_t is a $(nx \times 1)$ vector of *transformed* transition variables, related to actual transition $\mathbf{X_t}$ variables by $\mathbb{U}$ and ε_t is a vector of $(ne \times 1)$ errors. Matrices D, K are constants. We omit exogenous regressors in this note.

Transition dynamics is allowed to be non-stationary and diffuse initialization of $\mathbf{X_t}$, hence α_t must be adopted. In the current setting the assumption is that initial value of α_0 is unknown deterministic vector and the approach of `Rosenberg (1973)` is followed.

Notation of the Code We follow the notation of the code with few exceptions. In the code the vector α is denoted as x. Furthermore there are following timing conventions

$$\texttt{x10} = x_{t,t-1} \qquad \texttt{x11} = x_{t,t} = x_t. \tag{4}$$

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1 Filtering and Smoothing Problem

1.1 Initialisation of the Filter

Using information about eigenvalues of the transition dynamics, non-stationary components are identified (`ixunit`) and the variance of those is set to zero.

For stationary part of the transition vector the initial covariance matrix $P_{0,0}$ is initialised using unconditional variance of stationary variables.[1] The stationary partition of initial vector $x_{0,0}$ is set to its unconditional mean

$$x_{0,0}^{unfixed} = (I-T)^{-1}K. \tag{5}$$

The non-stationary partition of $x_{0,0}$ is set to zero, similarly to $P_{0,0}$.

The approach of `Rosenberg (1973)` makes use of the fact that the fixed-partition of $x_{0,0}$ is zero ($\tilde{x}_{0,0} = 0$) and thus may be written as

$$x_{0,0} = \tilde{x}_{0,0} + x_{0,0}. \tag{6}$$

Since (6) is linear, applying the Kalman filter to it results in a filtered values being linear combination of the data and the unknown initial state vector $x_{0,0}$.

For example, we have

$$x_{1,0} = Tx_{0,0} = T\tilde{x}_{0,0} + Tx_{0,0} \tag{7}$$

or for $t = 1,\ldots,T-1$

$$x_{t+1,t} = \tilde{x}_{t+1,t} + Q_t x_{0,0}, \tag{8}$$

where Q_t expresses accumulated impact of the initial unknown part of the state vector on period t value of filtered variables. Analogously we can express prediction errors. Setting up the likelihood and optimising w.r.t. unknown partition of $x_{0,0}$ we back out an estimate.

Hence, the standard Kalman filter method is applied to the initial state, where stationary partion is at its unconditional mean and non-stationary is zero. The recursion for Q_t is also calculated. After the estimate $\hat{x}_{0,0}$ is obtained, using again the linearity of the relations above a *correction* for effect of estimated initial conditions is carried out. Note that the Rosenberg approach is virtually identical to GLS estimation of the unknown initial state.

[1] The code does not partition the transition vector and uses `ixfixed` (based on `ixunit` to select appropriate part of matrices to base calculation on.

1.2 Kalman Filter

Having initialised the filtering problem, standard KF method is applied. The system of Kalman recursion is as follows

$$x_{t,t-1} = Tx_{t-1} + K \tag{9}$$
$$Y_{t,t-1} = Zx_{t,t-1} + D \tag{10}$$
$$v_t = Y_t - Y_{t,t-1} \tag{11}$$
$$x_{t,t} = x_{t,t-1} + P_{t,t-1}Z'F^{-1}v_t \tag{12}$$
$$P_{t,t-1} = TP_{t,t}T' + R\Omega R' \tag{13}$$
$$P_{t,t} = P_{t,t-1} - P_{t,t-1}Z'F_t^{-1}ZP_{t,t-1} \tag{14}$$
$$F_t = ZP_{t,t-1}Z' + H\Omega H'. \tag{15}$$

Note that recursions (15), (14) and (13) are data independent and may be ran separately. Many other terms –matrix multiplications and divisions– may also be calculated separately.

There is an additional recursion for the Q_t – the effect of unknown initial conditions:

$$Q_t = (T - TP_{t,t-1}Z'F^{-1}Z)Q_{t-1} \qquad Q_0 = T. \tag{16}$$

The estimate of the unknown initial condition –see e.g. `Harvey (1989)`– may be written as

$$\hat{x}_{0,0} = \left[\sum_{t=1}^{T} G'_{t-1}Z'F_t^{-1}ZG_{t-1}\right]^{-1} \sum_{t=1}^{T} G'_{t-1}Z'F_t^{-1}v_t^*. \tag{17}$$

Hence, we have

$$\hat{x}_{0,0} = U^{-1}V, \tag{18}$$

where

$$U_t = U_{t-1} + Q'_{t-1}Z'F_t^{-1}ZQ_{t-1} \tag{19}$$
$$V_t = V_{t-1} + Q'_{t-1}Z'F_t^{-1}v_t. \tag{20}$$

Correction If there are actually some non-stationary transition variables or variables fixed from other reasons a correction for estimated initial conditions $\hat{x}_{0,0}$ must be carried out, using Q_t.

Thus we have corrected filtered values

$$x^c_{t,t-1} = x_{t,t-1} + Q_t\hat{x}_{0,0} \tag{21}$$
$$Y^c_{t,t-1} = Y_{t,t-1} + ZQ_t\hat{x}_{0,0} \tag{22}$$
$$v^c_t = Y_t - Y^c_{t,t-1} \tag{23}$$
$$F_t^{-1}v_t^c = F_t^{-1}v_t - F_tZQ_t\hat{x}_{0,0} \tag{24}$$

1.3 Kalman Smoother

The Kalman smoother problem is standard and makes use of the filtered quantities, including the correction if necessary – in case of fixed unknown initial state.

The smoothing recursions may be written as

$$\begin{aligned}
\varepsilon^{ME}_{t|T} &= \Omega^{ME} H' \left[F_t^{-1} v_t - F_t^{-1} Z (T P_{t,t-1})\, b_t\right] && (25)\\
b_{t-1} &= Z' F_t^{-1} v_t + M_t' b_t && (26)\\
\varepsilon^{TE}_{t|T} &= \Omega^{TE} R' b && (27)\\
N &= Z' F_t^{-1} Z + M_t' N_t M_t && (28)\\
x_{t,T} &= x_{t,t-1} + P_{t,t-1} b_{t-1} && (29)\\
P_{t|T} &= P_{t,t-1} - P_{t,t-1} N_{t-1} P_{t,t-1} && (30)\\
M_t &= T - T(P_{t,t-1} Z' F_t^{-1} Z) = T - K_t Z && (31)\\
& b_T = N_T = 0. && (32)
\end{aligned}$$

New auxilliary variables b_t and N_t are used for backward smoothing. Note that K_t is the Kalman gain.[2]

[2]In Durbin and Koopman (2001) notation we have: $b \equiv r$, $N \equiv N$ and $M \equiv L$.