Likelihood-based Estimation of Stochastically Singular DSGE Models using Dimensionality Reduction

Michal Andrle[1]

Computing in Economics and Finance, Prague June 2012

[1] The views expressed herein are those of the author and should not be attributed to the International Monetary Fund, its Executive Board, or its management.

Outline of the Talk

- Motivation
- Stochastic Singularity
  - Definition & Problem
  - Data & Models
- Estimating Dimensionality reduction
  - Principal Components Transform
  - Dynamic Principal Components Transform
- Examples...

Motivation

- How many shocks do we need to explain an economy's dynamics?
- Many dynamic economic models feature only few truly structural shocks, yet have implications for a handful of macro variables
- Macroeconomic data feature a great amount of regularity
- Adding back newly defined 'regression error' to DSGE models

Stochastic Singularity – Definition

Assume a model is cast in terms of linear, time invariant state-space form

$$\mathbf{Y}_t = \mathbf{Z}\mathbf{X}_t \quad (1)$$
$$\mathbf{X}_t = \mathbf{T}\mathbf{X}_{t-1} + \mathbf{R}\varepsilon_t, \quad (2)$$

The model is 'stochastically singular' if a spectral density of observed variables $S_{yy}(\lambda)$ is rank-defficient at almost all frequencies.

Implications:

- Number of shocks is less than or equal to number of observables (sufficient condition)
- A combination of elements in **Y** is perfectly correlated.

Stochastic Singularity – Problem

- Macroeconomic and financial data are never perfectly correlated
- The model cannot be exact data-generating process for real data
- It precludes use of likelihood-based methods and Kalman filter
- Bayesian-likelihood methods are of no help here, of course ;)

Stochastic Singularity – Common Solutions (I)

A. Methods of moments (Ruge-Murcia, 2004)
   - What moment restriction are suitable?
   - How to estimate structural shocks and unobserved variables?

B. Restricting the set of observed variables
   - Loss of valuable information. Which variables to retain?

C. Introducing more ‘structural shocks’
   - Are many shocks used in the literature really structural?

Stochastic Singularity – Common Solutions (II)

D. Adding uncorrelated measurement error (Altug 1989, Sargent 1989)
   - $\mathbf{Y}_t = \mathbf{Z}\mathbf{X}_t + \mathbf{K}\nu_t$
   - Cross-correlations driven only by the model
   - Restrictive specification, misspecification issues
   - What variables are noisy?

E. Adding dynamic measurement error process (Ireland,2004)
   - $\mathbf{Y}_t = \mathbf{Z}\mathbf{X}_t + \mathbf{D}(\mathbf{L})\nu_t$
   - Increases the complexity and number of parameters of the model
   - Ad-hoc dynamics may overrule the structural model

Appeal of Stochastic Singularity

Models:

- Theory underpins only few truly structural driving forces
- Sharper economic identification of structural shocks

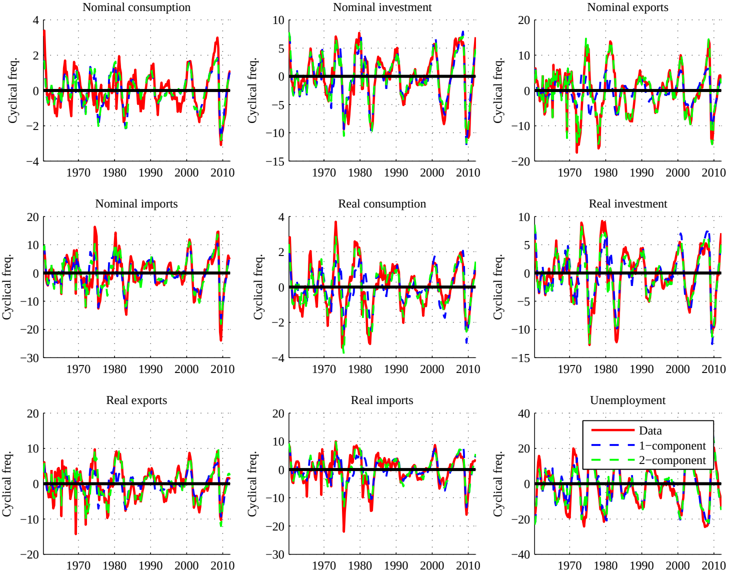
Data:

- Macroeconomic data feature robust regularities and co-movement
- At business cycle frequency, real economic variables are driven by few factors (Andrle and Brůha, 2012)
- Real and nominal cyclical comovement is very strong (Andrle, 2012)

Real Comovements – Principal Components Analysis

United States. . .

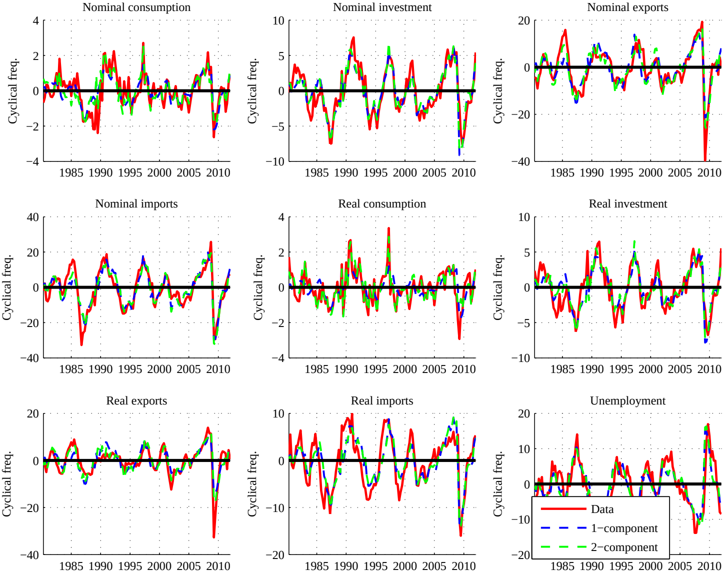
Source: Andrle and Brůha (2012)



Real Comovements – Principal Components Analysis

Japan. . .

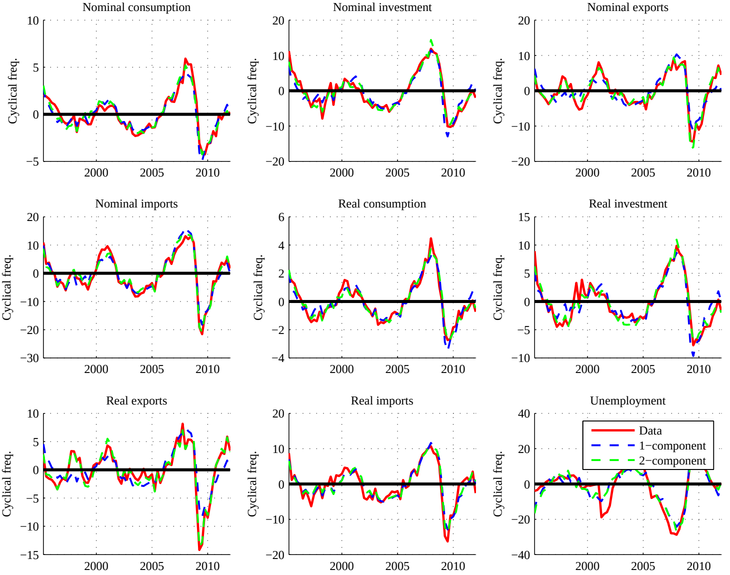
Source: Andrle and Brůha (2012)



Real Comovements – Principal Components Analysis

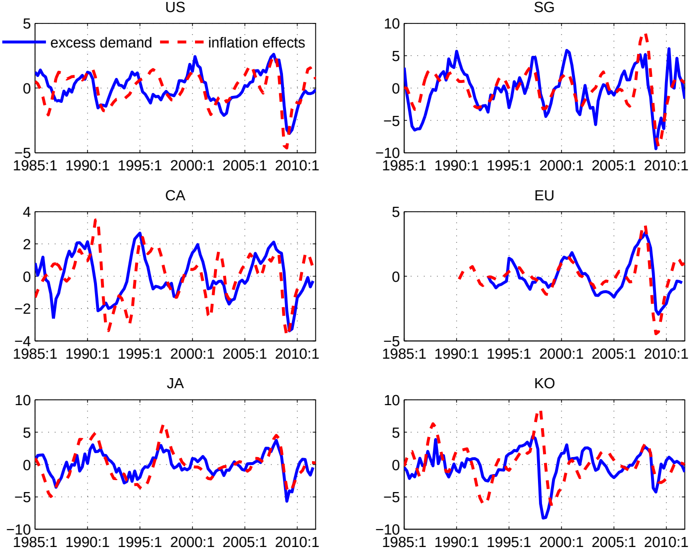
Spain. . .

Source: Andrle and Brůha (2012)



Real and Nominal Comovements

Source: Andrle (2012)



New Estimation Method for Singular Models

Our models and estimation method should exploit data regularities!

Underlying intuitive approach:

1. Find the principal components and principal subspace of the data (subspace of 'maximum covariance')
2. Use at most n_e dimensions, where n_e is number of shocks
3. Rotate the model into this subspace
4. Estimate transformed model using likelihood-based methods
5. Estimate (filter) unobserved variables and structural shocks

The estimator penalizes models incompatible with robust feature of the data.

Principal Components – Static vs. Dynamic

Stochastic singularity relates to dynamic rank of the model/data.

Consider a well-known example, a simple process $\mathbf{U}(t)$ driven by serially uncorrelated stationary process $\nu(t) \sim N(0, \sigma^2)$

$$\mathbf{U}_t = \begin{bmatrix} \nu_t \\ \alpha\nu_{t-1} \end{bmatrix}, \ \Gamma_U^0 = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \alpha^2\sigma^2 \end{bmatrix}, \ \Gamma_U^{+1} = \begin{bmatrix} 0 & 0 \\ \alpha\sigma^2 & 0 \end{bmatrix}, \qquad (3)$$

with rank-defficient spectral density $\mathbf{S}_U(\lambda)$

$$\mathbf{S}_U(\lambda) = \begin{bmatrix} 1 & \alpha e^{-i\lambda} \\ \alpha e^{i\lambda} & \alpha^2 \end{bmatrix}. \qquad (4)$$

Static rank (PCA) is 2.
Dynamic rank (DPCA) is 1 – only one driving force.

PC-MLE – Classic Principal Components (i)

Let $\Sigma_{\mathcal{Y}}$ be a covariance matrix of observed data $\mathcal{Y}$.

Using principal component analysis find $\mathbf{P}$ and Λ

$$\Sigma_{\mathcal{Y}} = \mathbf{P}\Lambda\mathbf{P}' \qquad r = \text{rank}(\Sigma_{\mathcal{Y}}), \tag{5}$$

where P, Λ denote matrix of eigenvectors and diagonal matrix of eigenvalues, respectively.

Define $\mathbf{P}^k$ to be a projection matrix on a subspace spanned by $k \leq n_e \leq r$ first principal components of the data, hence

$$\mathbf{F}_t^k = \mathbf{P}^k \mathcal{Y}_t, \qquad \hat{\mathcal{Y}}_t = \bar{\mathbf{P}}^k \mathbf{F}_t^k = \bar{\mathbf{P}}^k \mathbf{P}^k \mathcal{Y}_t \tag{6}$$

where $\mathbf{F}_t^k$ is a $(k \times 1)$ vector and $\mathbf{P}^k$ is a $(k \times n_y)$ projection matrix. $\hat{\mathcal{Y}}_t$ is a 'recovered' signal based on low-rank dimensionality transform.

PC-MLE – Classic Principal Components (ii)

The approximation error determines the measurement error process:

$$\mathbf{e}_t = \mathcal{Y}_t - \hat{\mathcal{Y}}_t \tag{7}$$

with a spectral density $\mathbf{S}_e(\lambda) = \mathbf{L}\mathbf{S}_{\mathcal{Y}}(\lambda)\mathbf{L}'$ where $\mathbf{L} = (\mathbf{I} - \bar{\mathbf{P}}^k\mathbf{P}^k)$.

Transform the model – project it into factor space:

$$\mathbf{F}^k_{S,t} = \mathbf{K}\mathbf{X}_t \tag{8}$$

$$\mathbf{X}_t = \mathbf{T}\mathbf{X}_{t-1} + \mathbf{R}\varepsilon_t, \tag{9}$$

where $\mathbf{F}^k_{m,t} \equiv \mathbf{P}^k\mathbf{Y}_t$, with $\dim(F_{S,t}) = k \leq n_e$, and $\mathbf{K} = \mathbf{P}^k\mathbf{Z}$, which is $(k \times n_x)$.

Conditional on $\mathbf{P}^k$, form standard log-likelihood criterion and use conventional Kalman filter.

DPC-MLE – Dynamic Principal Components (i)

Dynamic Principal Components – David Brillinger (1981, Ch. 9)

The problem is to determine $(k \times n_y)$ linear filter $\mathbf{P}^k(L)$ and $(n_y \times k)$ filter $\bar{\mathbf{P}}^k(L)$ such as to minimize

$$\mathbb{E}||\mathcal{E}|| = \mathbb{E}||\mathcal{Y}_t - \hat{\mathcal{Y}}_t|| = \mathbb{E}||\mathcal{Y}_t - \bar{\mathbf{P}}^k(L)\mathbf{P}^k(L)\mathcal{Y}_t|| \tag{10}$$

Boils down to PCA on spectral density at each frequency.

Data pre-filtering using DPCA filter determines the measurement error process:

$$\mathbf{S}_{\mathcal{E}}(\lambda) = [I - \mathbf{A}(\lambda)]\mathbf{S}_{\mathcal{Y}}(\lambda)[I - \mathbf{A}(\lambda)]^H, \tag{11}$$

where $\mathbf{A}(\lambda) = \bar{\mathbf{P}}^k(\lambda)\mathbf{P}^k(\lambda)$ is a 'purifying filter'.

DPC-MLE – Dynamic Principal Components (ii)

Denote $\mathbf{F}_{D,t}$ be the vector of dynamic principal components using DPCA filter $\mathbf{P}^k(z)$ if

$$\mathbf{F}^k_{D,t} = \mathbf{P}^k(L)\mathcal{Y}_t \qquad \hat{\mathcal{Y}} = \bar{\mathbf{P}}^k(L)\mathbf{F}^k_{D,t} \qquad \mathbf{P}^k(z) = \sum_{i=-\infty}^{\infty} \mathbf{P}^k_i z^i. \quad (12)$$

Complication: $\mathbf{P}^k(z)$ is possibly infinite!

The transformation of the model and estimation is carried out in frequency-domain, where the problem is well defined.

DPC-MLE – Dynamic Principal Components (iii)

Rotate (transform) the model into DPC space:

$$\mathbf{F}^k_{D,t} = \mathbf{P}^k(L)\mathbf{Z}\mathbf{X}_t \tag{13}$$

$$\mathbf{X}_t = \mathbf{T}\mathbf{X}_{t-1} + \mathbf{R}\varepsilon_t. \tag{14}$$

Formulate Whittle likelihood

$$\log \mathcal{L} \propto -\frac{1}{2}\sum_{j=0}^{T-1} \log[\det \mathbf{S}_F(\lambda_j)] \tag{15}$$

$$-\pi \times \mathrm{tr} \sum_{j=0}^{T-1} [\mathbf{S}_F(\lambda_j)^{-1} \mathbf{I}_{\hat{\mathcal{F}_{\mathcal{D}}}}(\lambda_j)],$$

Very efficient & fast way of evaluating likelihood and identification checks.

DPC-MLE – Dynamic Principal Components (iv)

Structural shocks cannot be estimated using the Kalman filter.

Use a Wiener-Kolmogorov filter implied by the transformed model:

$$\hat{\mathbf{X}}_{t|T} = \mathbf{W}(L)\mathbf{P}^k(L)\mathcal{Y}_t \qquad \mathbf{W}(z) = \mathbf{S}_{X,F_D}(z)\mathbf{S}_{F_D}(z)^{-1}, \tag{16}$$

where $\mathbf{S}_{X,F_D}(\lambda)$ is a cross-spectral density matrix and $\mathbf{S}_{F_D}(\lambda)$ is a spectral density matrix implied by the parameterized infinite-dimensional state-space model

DPC-MLE – Dynamic Principal Components (v)

Some important questions & considerations:

- Principal component estimates are not scale invariant.
- Should one compute componets on data or of the model?
- 'Good' estimates of sample spectral density are not easy to get

Simple Example (I.)

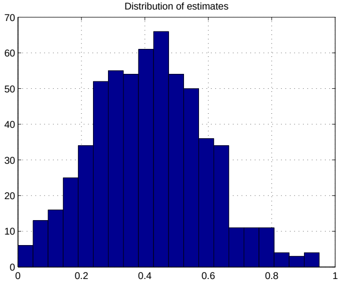
Simple model:

$$x_t = \rho_x x_{t-1} + \varepsilon_x \tag{17}$$

$$u_t = \rho_u u_{t-1} + (1-\rho_u)\alpha_{x,u} x_t + \varepsilon_u, \tag{18}$$

Parameterized: $\rho_x = 0.90, \rho_u = 0.4, \alpha_{x,u} = 0.4, \sigma_x = \sigma_u = 0.1$

Observed u_t, T=200, N=600 replications.



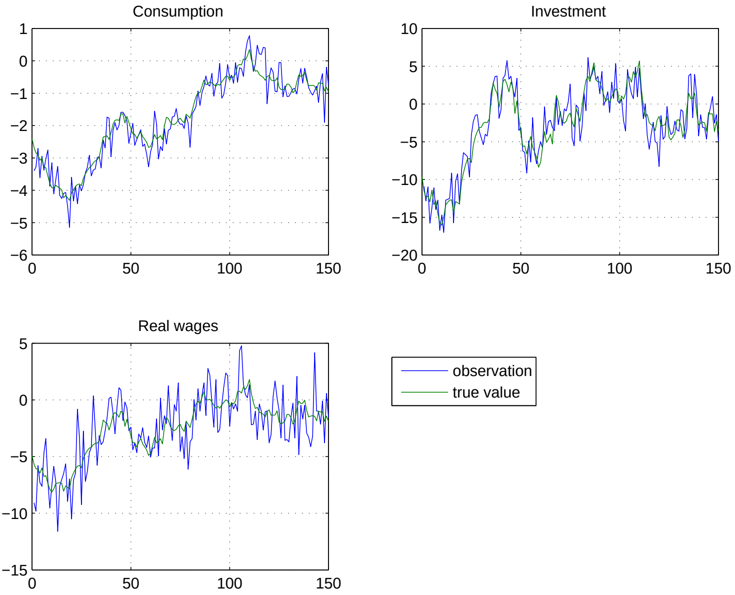
RBC Example – One Component/Factor (i)

Simple, standard RBC model:

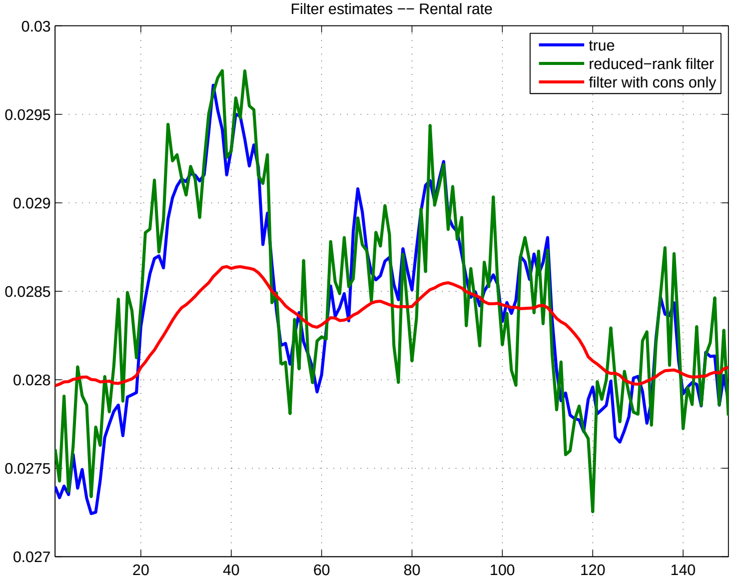
- Single structural shock (technology)
- Measurement errors added. . .

- One component/factor computed using cons, inv & wages
- Can we recover the unobserved components well?

RBC Example – One Component/Factor (ii)

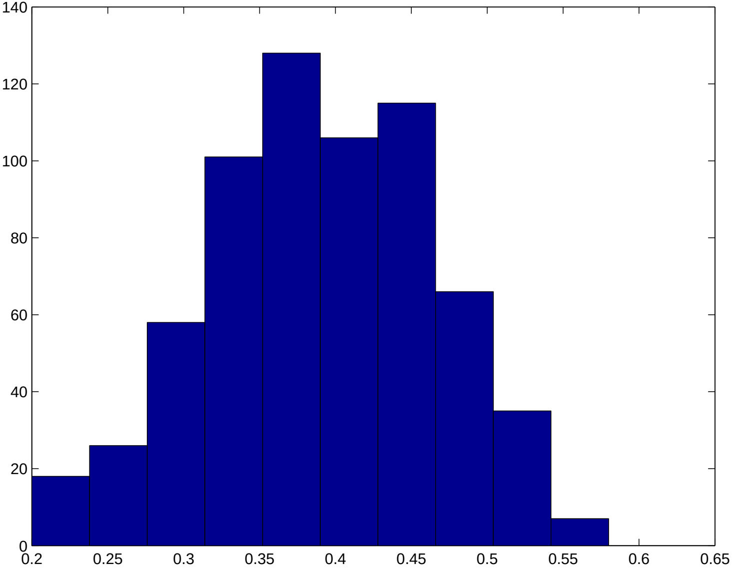


RBC Example – One Component/Factor (iii)



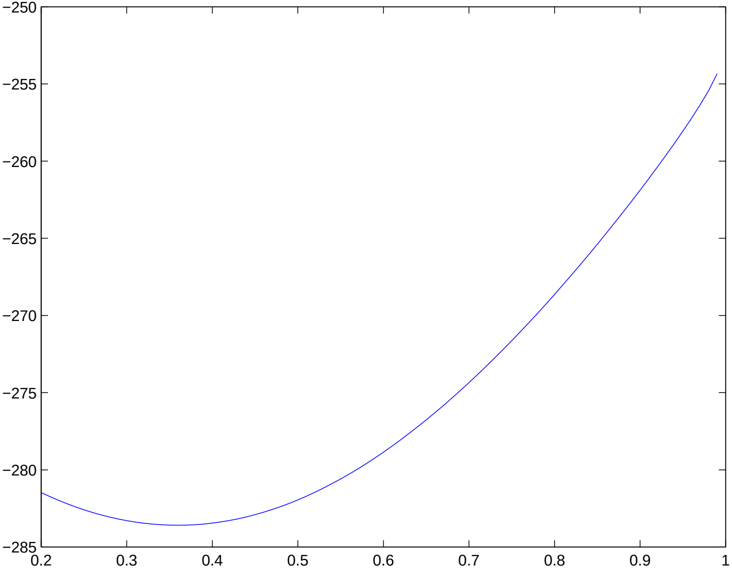
RBC Example – One Component/Factor (iv)

Histogram of estimates for ρ



RBC Example – One Component/Factor (v)

Identification/sensitivity for ρ



Issues & Caveats

- The best thing is to incorporate most of data considerations directly to the model
- There are many reasons not to base estimation on likelihood, or unconditional moments only in general...
- Likelihood assumes your model is a correct one, pure likelihood methods are not very robust

Conclusions

- Stochastic singularity can be a useful thing in a model
- Thinking about the data, number of shocks and ‘how much’ one wants to ‘fit’ is useful
- Dimensionality reduction methods provide one way how to extract important features in the data and comovements

Thank you for your patience…